Quantum Fourier Transforms

Special Topics in Computer Science:
Quantum Computing

CSC591/ECE592 – Fall 2019

Outline

- Math review – complex roots of unity
- Introduction concept of Fourier transforms for continuous functions
- Applying Fourier transforms to digital computers
- Discrete Fourier Transform
- Mapping Discrete Fourier Transform to Quantum Fourier Transform
- Derivation of formula for generalized Quantum Fourier Transform
- Worked example Quantum Fourier Transform for 3 qubits
- Mapping 3 qubit Quantum Fourier Transform to quantum computing gates

Math Review - Roots of Unity

- Let $\omega_N \equiv e^{2\pi i/N}$
- For a given N generate all possible values

$$\{1, \omega^1, \omega^2\ , \omega^3\ \ldots\ldots\ldots \omega^{N-1}\}$$

$$\{1, \omega^{e^{2\pi i/N}}, \omega^{e^{4\pi i/N}}\ , \omega^{e^{6\pi i/N}}\ \ldots\ldots\ldots \omega^{(N-1)e^{2\pi i/N}}\}$$

- Replace the continuous version of x, e^{isx} parameterized by s) into N vectors

$$\mathbf{v}_{\mathrm{j}} = \begin{pmatrix} v_{j0} \\ v_{j1} \\ \vdots \\ v_{jk} \end{pmatrix} = \begin{pmatrix} \omega^{-j0} \\ \omega^{-j1} \\ \vdots \\ \omega^{-jk} \end{pmatrix}$$

where k is the coordinate index and j is the parameter that labels each vector

Nth Roots of Unity

- Begin with the Taylor series expansion of the exponential function

$$\omega_N = \exp\left(\frac{2\pi i}{N}\right) = \cos\left(\frac{2\pi}{N}\right) + i\sin\left(\frac{2\pi}{N}\right)$$

- Compute the unit modulus

$$|\omega_N|^2 = \omega_N^*\omega_N = \exp\left(-\frac{2\pi i}{N}\right)\exp\left(\frac{2\pi i}{N}\right)\exp\left(-\frac{2\pi i}{N} + \frac{2\pi i}{n}\right)$$

$$= \exp(0) = 1$$

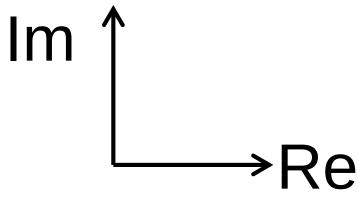
- Make a graphical construction of these N roots of unity

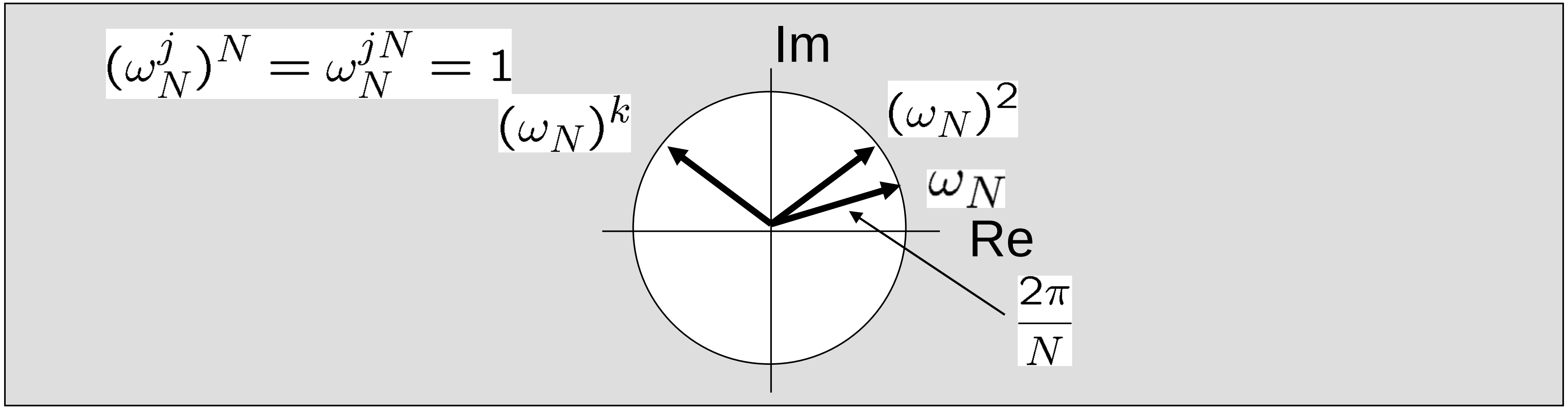
Graphical Construction of These N Roots of Unity

The sum of all sums:

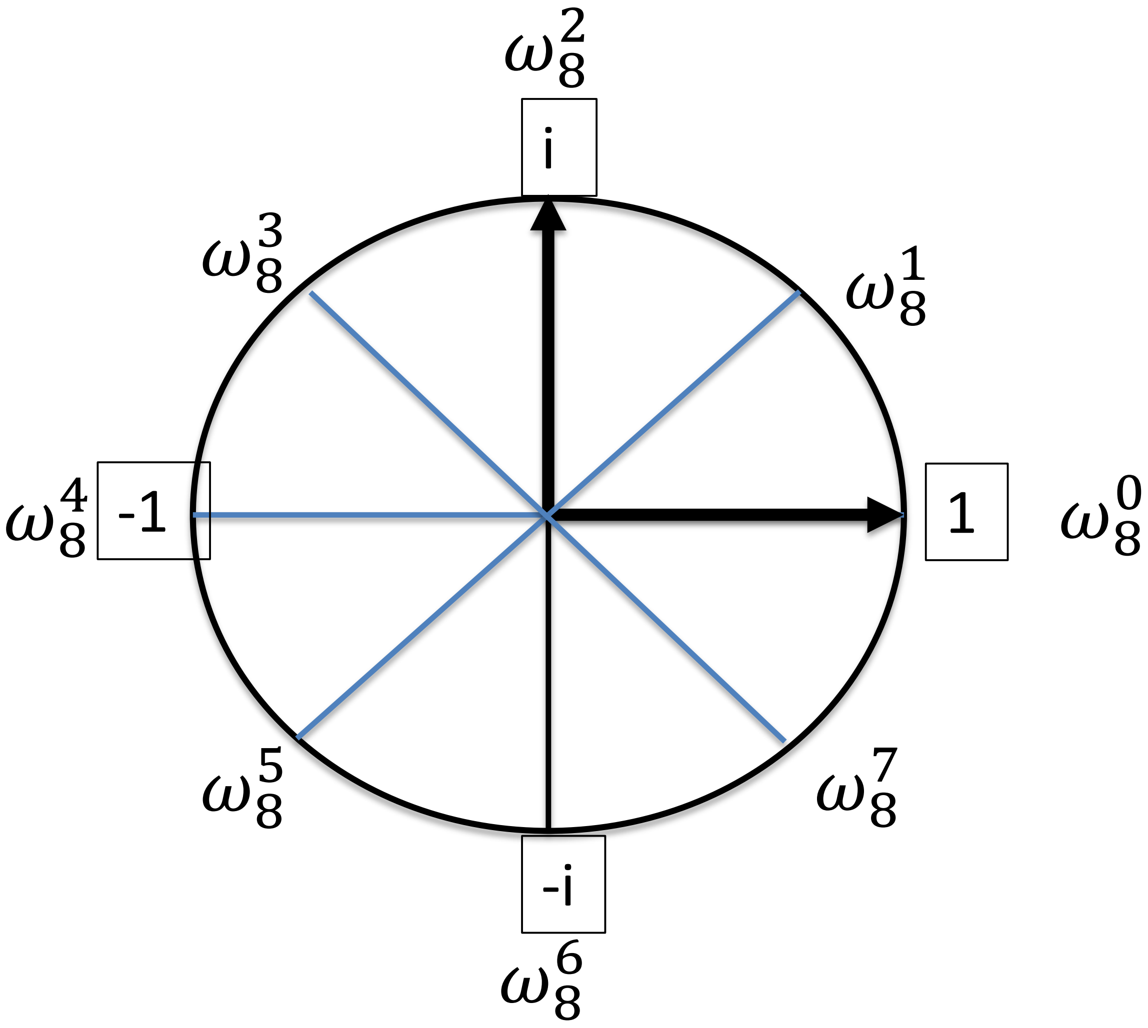
$$\frac{1}{N}\sum_{k=0}^{N-1}(\omega_N)^{jk} = \delta_{j,0}$$

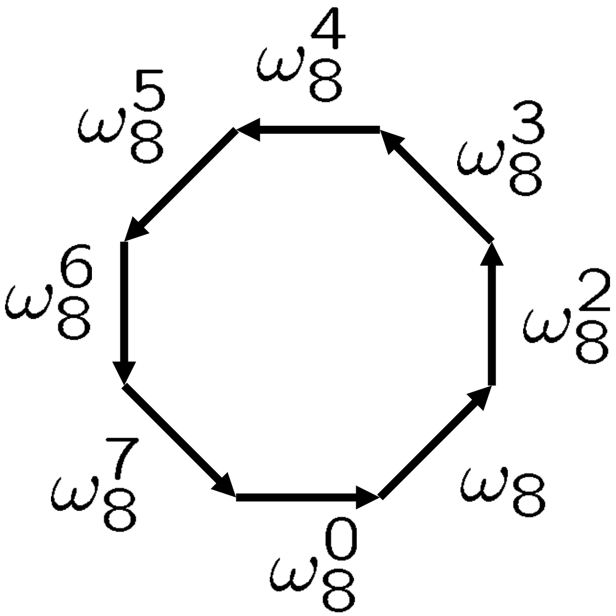
Construct a coordinate system in the complex plane with real and imaginary axes





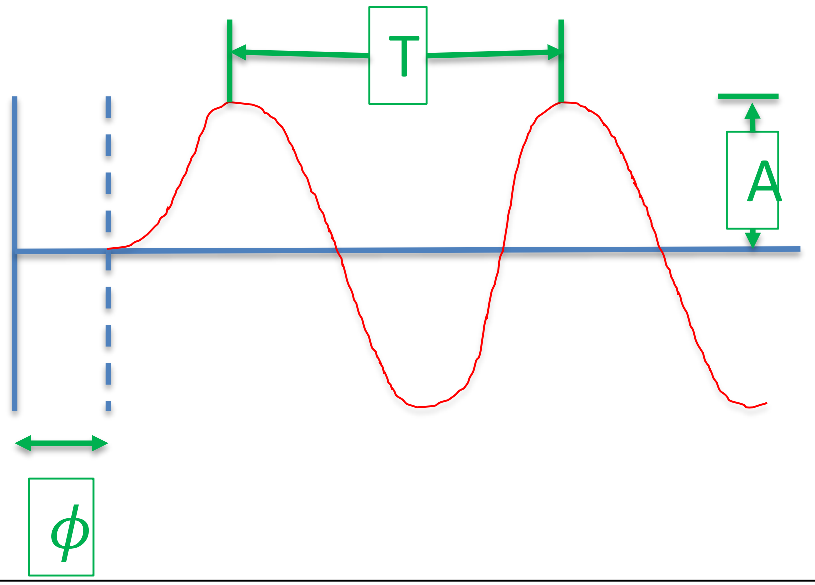
Example 8th Roots of Unity Plotted on Complex Plane





Fourier Transforms

- FT is a mapping between two domains
  - Time and frequency
  - position and momentum
- Can combine many different signals each with their own frequency, amplitude and phase



Fourier Transform for Continuous Functions

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{ikx} dk$$

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

Generic Expression for Fourier Transform and Inverse Fourier Transform

$$F(s) \quad = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx \ .$$

$$f(x) \quad = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{isx}\, ds \ .$$

Application of Fourier Transform Ideas to Digital Computers

- Continuous mathematical formulations are incompatible with digital or quantum computers
- Computers are discrete and finite collections of bits (or qubits)
- Need to modify the continuous Fourier Transform to a digital formulation
- This digital formulation needs to ultimately be integrated into the hardware architecture of a quantum computer
- To make this happen it means designing and building gate operations that can be understood by a quantum computer

Discrete Fourier Transform (DFT)

- Need to construct an equivalent summation expression that can reference the Discrete Fourier Transform's functionality versus the continuous Fourier Transform

- **Definition:** An n^{th} order DFT is a function of an n-component vector f = (f_k) that produces a new n-component vector F = (F_j) given by a formula that describes the output vector's n-components

$$F_j \equiv \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} f_k \omega^{-jk} \qquad j=0,\ 1,\ldots.N-1$$

- {F_j} provides the "weighting factors" for the expansion of f as a weighted sum of the frequencies ω^j

$$f_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} F_j \omega^{jk}$$

- Essentially the real functions that were used in the continuous Fourier transforms have become complex roots of unity in the discrete Fourier transform

Matrix Representation of a Discrete Fourier Transform

NOTATION $\alpha \equiv \omega^{-1}$ AND $\omega = \omega_N = \exp\left(\frac{2\pi i}{N}\right)$

- THIS MATRIX ENCODES THE DFT FUNCTIONALITY

$$Z = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \alpha & & & \alpha^{N-1} \\ 1 & \alpha^{j} & \alpha^{2j} & & \alpha^{(N-1)j} \\ \vdots & & & & \\ 1 & \alpha^{N-1} & \alpha^{2(N-1)} & \cdots & \alpha^{(N-1)(N-1)} \end{pmatrix}$$

- THE DFT OF THE VECTOR f_k CAN BE WRITTEN IN MATRIX FORM

$$\frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \alpha & \alpha^{2} & \alpha^{3} & & \alpha^{N-1} \\ 1 & \alpha^{2} & \alpha^{4} & \alpha^{6} & & \alpha^{2(N-1)} \\ \vdots & & \vdots & \vdots & & \vdots \\ 1 & \alpha^{j} & \alpha^{2j} & \alpha^{3j} & & \alpha^{j(N-1)} \\ \vdots & & \vdots & \vdots & & \vdots \\ 1 & \alpha^{N-1} & \alpha^{2(N-1)} & \alpha^{3(N-1)} & & \alpha^{(N-1)(N-1)} \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_k \\ \vdots \\ f_{N-1} \end{pmatrix}$$

Consequences of Transformation from Continuous Fourier Transform to Discrete Fourier transforms

- Three consequences
  1. Integrals become sums from 0 to N-1. [Rather than evaluating f at real continuous x the evaluation is over a complex spectrum vector F_j that also has N components
  2. The factor $\frac{1}{2\pi}$ is replaced by a normalizing factor $\frac{1}{\sqrt{N}}$

     NOTE: This choice is driven by the need for all vectors to live in the projective sphere in Hilbert space and therefore be normalized
  3. The complex roots of unity ω^{jk} replace the general exponential e^{isx}

SUMMARY - DFT + QFT

$|x\rangle = |x_{m-1}, x_{m-2} \ldots x_1 x_0\rangle$ basis

$$U|x\rangle = \sum_{y=0}^{N-1} \underbrace{|y\rangle\langle y|}_{\text{INSERT COMPLETE SET OF STATES}} \underbrace{U|x\rangle}_{\text{MATRIX ELEMENT}}$$

$$= \sum_{y=0}^{N-1} U(y,x)\,|y\rangle \qquad \left[\tilde{f}(y) = \sum_{x=0}^{N-1} K(y,x)\,|x\rangle \right.$$

FROM DFT $\left.\right]$

$$U \sum_x F(x)\,|x\rangle = \sum_x F(x)\,U|x\rangle$$

$$= \sum_x F(x) \sum_y K(x,y)\,|y\rangle$$

$$= \sum_y \underbrace{\left(\sum_x K(x,y)\,F(x) \right)}_{\text{INVERSE FOURIER TRANSFORM}} |y\rangle$$

$$= \sum_y \tilde{f}(y)\,|y\rangle$$

Transforming from Discrete Fourier Transform to Quantum Fourier Transform

- Defined the Discrete Fourier Transform that a given vector $x \in \mathbb{C}^N$ outputs another vector $y \in \mathbb{C}^N$ such that

$$y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j \omega^{jk}$$

- The Quantum Fourier Transform transforms a basis set {|0>, |1>, …{N-1>} into another basis set such that

$$|\mathrm{j} >= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} |k > \omega^{jk}$$

Matrix Representation of a Quantum Fourier Transform

$$QFT_N = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{(N-1)} \\ 1 & \omega^2 & \omega^4 & \omega^6 & & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \omega^{3(N-1)} & \cdots & \omega^{(N-1)(N-1)} \end{pmatrix}$$

- ENTRIES IN THE MATRIX ARE THE N^{th} ROOTS OF UNITY

$$\omega = e^{\frac{2\pi i}{N}} = \cos\left(\frac{2\pi}{N}\right) + i \sin\left(\frac{2\pi}{N}\right)$$

Ex $N=8$

$$\omega = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i = \frac{1}{\sqrt{2}}(1+i)$$

- MATRIX CAN BE REPRESENTED AS

$$\begin{array}{c|c} & 0 \quad 1 \quad \cdots \quad k \quad \cdots \quad N-1 \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ j \\ \vdots \\ N-1 \end{array} & \begin{pmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & \omega^{jk} & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

Derivation of the Recursion Relation for N^{th} Order QFT

Notation

QFT CIRCUIT

NOTATION

- $|x\rangle^m = |x_{m-1}\rangle \otimes |x_{m-2}\rangle \otimes \ldots\ldots \otimes |x_0\rangle = \bigotimes_{k=0}^{m-1} |x_k\rangle$

$|x_k\rangle$ CAN TAKE VALUES $|0\rangle$ OR $|1\rangle$

- THIS IS A DELIBERATE CHOICE TO DECREASE ORDER FROM x_{m-1} TO x_0 $\Rightarrow$ WANT THE RIGHT-MOST BIT TO CORRESPOND TO THE LEAST SIGNIFICANT BIT OF THE BINARY NUMBER $x_{m-1} \ldots x_1 x_0$ IS DECIMAL

- ALTERNATE REPRESENTATION

$$\left.\begin{array}{lcl} |0\rangle^4 & \longleftrightarrow & |0000\rangle \\ |1\rangle^4 & \longleftrightarrow & |0001\rangle \\ |2\rangle^4 & \longleftrightarrow & |0010\rangle \\ \vdots & & \\ |15\rangle^4 & \longleftrightarrow & |1111\rangle \end{array}\right\} \quad x = \sum_{k=0}^{m-1} x_k 2^k$$

Sample Derivation for N=3

N = 3 EXAMPLE ILLUSTRATION

$$\sqrt{N}\ \mathrm{QFT}|x\rangle^{n} = \sum_{y=0}^{N-1} \phi_{xy} |y_{m-1} y_{m-2} \cdots y_1 y_0\rangle$$

$$\sqrt{8}\ \mathrm{QFT}|x\rangle^{3} = \phi_{x0}|000\rangle + \phi_{x1}|001\rangle + \phi_{x2}|010\rangle + \phi_{x3}|011\rangle$$
$$+ \phi_{x4}|100\rangle + \phi_{x5}|101\rangle + \phi_{x6}|110\rangle + \phi_{x7}|111\rangle$$

REGROUP INTO EVEN-ODD

$$= \phi_{x0}|000\rangle + \phi_{x2}|010\rangle + \phi_{x4}|100\rangle + \phi_{x6}|110\rangle$$
$$+ \phi_{x1}|001\rangle + \phi_{x3}|011\rangle + \phi_{x5}|101\rangle + \phi_{x7}|111\rangle$$

= y-even group Σ

\+ y-odd group Σ

Group Even Terms and Factor Right-Most Bit

N = 3 EXAMPLE   Y-EVEN GROUPING

- START WITH CASE $y_0 = 0$

$$\omega^{x y_0 2^0} = \omega^{x \cdot 0 \cdot 1} = 1$$

$$\Rightarrow \phi_{x0} = \prod_{k=0}^{2} \omega^{x y_k 2^k} = \prod_{k=1}^{2} \omega^{x y_k 2^k}$$

STARTS AT $k=1$ BECAUSE THE $k=0$ FACTOR IS "1"

$$y \text{ EVEN GROUP} = \phi_{x0}|000\rangle + \phi_{x2}|010\rangle + \phi_{x4}|100\rangle + \phi_{x6}|110\rangle$$

FACTOR OUT THE $|y_0\rangle = |0\rangle$ $\Rightarrow$

$$= \left[\phi_{x0}|00\rangle + \phi_{x2}|01\rangle + \phi_{x4}|10\rangle + \phi_{x6}|11\rangle\right]|0\rangle$$

$$= \left[\sum_{\substack{y=0 \\ y\text{-even}}}^{7} \left(\prod_{k=1}^{2} \omega^{x y_k 2^k}\right)|y_2 y_1\rangle\right]|0\rangle$$

FACTORING OUT THE RIGHT MOST BIT LEADS TO SIMPLIFICATIONS

Group Even Terms and Factor Right-Most Bit (cont'd)

N = 3 EXAMPLE

QFT CIRCUIT - EVEN GROUPING

CONSEQUENCES OF FACTURING OUT RIGHT MOST BIT

1) $\sum_{y \text{ even}}^{7} \rightarrow \sum_{\text{all}}^{3}$ HALVES THE TOTAL SUM

2) $|y_2 y_1\rangle \rightarrow |y_1 y_0\rangle$

3) $\prod_{1}^{2} \rightarrow \prod_{0}^{1}$ SHIFTS THE PRODUCT FACTOR

4) $2^k \rightarrow 2^{k+1}$

FOLD ABOVE 4 BULLETS INTO THE y-EVEN PORTION OF QFT EQ.

$$y\text{-EVEN} = \left[\sum_{y=0}^{3} \left(\prod_{k=0}^{1} \omega^{x y_k 2^{k+1}}\right) |y_1 y_0\rangle\right] |0\rangle$$

$$= \left[\sum_{y=0}^{3} \left(\prod_{k=0}^{1} (\omega^2)^{x y_k 2^k}\right) |y_1 y_0\rangle\right] |0\rangle$$

Generalize to “N” Even Terms

QFT CIRCUIT

EVEN GROUPING

N = 3 EXAMPLE

- THE FACTOR x STILL RUNS OVER THE ORIGINAL RANGE
- NOTE WHEN x > 3 CAN REPLACE WITH "x − 4" BECAUSE ROOTS OF UNITY OBEY MODULAR ARITHMETIC "x MOD 4"

$$y\text{-EVEN} = \left[\sum_{y=0}^{3}\left(\prod_{k=0}^{1}(\omega^2)^{x y_k 2^k}\right)|y_1 y_0\rangle\right]|0\rangle$$

$$= \left[\sum_{y=0}^{3}\left(\prod_{k=0}^{1}(\omega^2)^{(x \bmod 4) y_k 2^k}\right)|y_1 y_0\rangle\right]|0\rangle$$

GENERALIZING TO ANY VALUE "n"

$$= \left[\sum_{y=0}^{\frac{N}{2}-1}\left(\prod_{k=0}^{n-2}(\omega^2)^{(x \bmod 4) y_k 2^k}\right)|y_{n-2}\, y_{n-3} \cdots y_0\rangle\right]|0\rangle$$

Group Odd Terms and Factor Right-Most Bit

QFT CIRCUIT

ODD GROUPING

N=3 EXAMPLE

START WITH CASE $y_0 = 1$

$$\omega^{x y_0 2^0} = \omega^{x(1)(1)} = \omega^x$$

$$\Rightarrow \phi_{xy} = \prod_{k=0}^{2} \omega^{x y_k 2^k} = \omega^x \prod_{k=1}^{2} \omega^{x y_k 2^k} \leftarrow$$

FACTORING OUT ω^x TERM ALLOWS THE PRODUCT TO START AT $k=1$

* ALIGNS WITH EVEN GROUP FORMULATION

$$y\text{-ODD GROUP} = \phi_{x1}|001\rangle + \phi_{x3}|011\rangle + \phi_{x5}|101\rangle + \phi_{x7}|111\rangle$$

$$= \left[\phi_{x1}|00\rangle + \phi_{x3}|01\rangle + \phi_{x5}|10\rangle + \phi_{x7}|11\rangle\right]|1\rangle$$

$$= \omega^x \left[\sum_{\substack{y=0 \\ y\text{-ODD}}}^{7} \left(\prod_{k=1}^{2} \omega^{x y_k 2^k}\right)|y_2 y_1\rangle\right]|1\rangle$$

Group Odd Terms and Factor Right-Most Bit (cont'd)

QFT CIRCUIT

y ODD GROUPING

CONSEQUENCES OF FACTORING OUT RIGHT MOST BIT

1.) $\sum_{ODD}^{7} \rightarrow \sum_{ALL}^{3}$ HALVES THE TOTAL SUM

2.) $|y_2 y_1\rangle \rightarrow |y_1 y_0\rangle$

3.) $\prod_{1}^{2} \rightarrow \prod_{0}^{1}$ SHIFTS THE PRODUCT FACTOR

4.) $2^k \rightarrow 2^{k+1}$

FOLD THE ABOVE 4 BULLETS INTO THE y-ODD PORTION OF THE QFT EQ

$$y\cdot ODD = \omega^x \left[\sum_{y=0}^{3} \left(\prod_{k=0}^{1} (\omega^2)^{x y_k 2^k} \right) |y_1 y_0\rangle \right] |1\rangle$$

Generalize to "N" Odd Terms

N=3 EXAMPLE

QFT CIRCUIT

ODD GROUPING

PERFORM SAME "x MOD 4" → x AS WAS DONE ON y-EVEN GROUPING

$$\text{"}y\text{-ODD"} = \omega^{x}\left[\sum_{y=0}^{3}\left(\prod_{k=0}^{1}(\omega^{2})^{(x \bmod 4)\, y_k 2^k}\right)|y_1 y_0\rangle\right]|1\rangle$$

FOR GENERAL n

$$= \omega^{x}\left[\sum_{y=0}^{N/2-1}\left(\prod_{k=0}^{n-2}(\omega^{2})^{(x \bmod \frac{N}{2})\, y_k 2^k}\right)|y_{n-2} y_{n-3} \cdots y_1 y_0\rangle\right]|1\rangle$$

Quantum Fourier Transform Recursion Relation

QFT CIRCUIT
RECURSION RELATION

REPEATING THIS PROCEDURE (RECURSIVELY)

$$QFT^{2^n}|x\rangle^n = \prod_{k=1}^{n}\left(\frac{|0\rangle + \omega_{2^k}^{x}|1\rangle}{\sqrt{2}}\right)$$

NOTE

RHS OF THE ABOVE EQ. IS EXPRESSED IN TERMS OF ROOTS OF UNITY

$\omega_2 \quad \omega_4 \quad \cdots \quad \omega_{2^n}$

WRITTEN IN POWERS OF $\omega = \omega_N$

$$\omega_{2^k} = \omega^{2^{n-k}}$$

$$QFT^{2^n}|x\rangle^n = \prod_{k=1}^{n}\left(\frac{|0\rangle + \omega^{2^{n-k}x}|1\rangle}{\sqrt{2}}\right)$$

Example of a 3 Qubit Quantum Fourier Transform

8^{th} Complex Roots of Unity

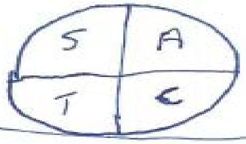
QFT-EX-0

COMPLEX ROOTS OF UNITY

EIGHT ROOTS OF UNITY

$$e^{k\left(\frac{2\pi i}{8}\right)} = e^{k\left(\frac{i\pi}{4}\right)} \qquad k = 0, \ldots 7$$

$$e^{x} = \cos x + i \sin x$$



k	
0	$e^{0\left(\frac{i\pi}{4}\right)} = e^{0} = \boxed{+1}$
1	$e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \boxed{\frac{1}{\sqrt{2}}(1+i)}$
2	$e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i = \boxed{i}$
3	$e^{3\left(\frac{i\pi}{4}\right)} = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) = \boxed{\frac{1}{\sqrt{2}}(-1+i)}$
4	$e^{4\left(\frac{i\pi}{4}\right)} = e^{+i\pi} = \cos(\pi) + i\sin(\pi) = \boxed{-1}$
5	$e^{5\left(\frac{i\pi}{4}\right)} = e^{\frac{5\pi i}{4}} = \cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right) = \boxed{-\frac{1}{\sqrt{2}}(+1+i)}$
6	$e^{6\left(\frac{i\pi}{4}\right)} = \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = \boxed{-i}$
7	$e^{7\left(\frac{i\pi}{4}\right)} = \cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right) = \boxed{\frac{1}{\sqrt{2}}(1-i)}$

General Formula for QFT and
First Factor for the 3 Qubit Example Construction

TRANSLATING THE MATH TO A QFT QUANTUM CIRCUIT

EXAMPLE $m=3$

$$QFT^{(8)}|x\rangle^3 = \left(\frac{|0\rangle + \omega^{4x}|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle + \omega^{2x}|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle + \omega^{x}|1\rangle}{\sqrt{2}}\right)$$

FIRST FACTOR

NOTE ω IS AN 8^{TH} ROOT OF UNITY $\Rightarrow$ COEFFICIENT OF $|1\rangle$ IN TERM $\left(\frac{|0\rangle + \omega^{4x}|1\rangle}{\sqrt{2}}\right)$ CAN BE DERIVED BY

$$\omega^{4x} = \omega^{4(4x_2 + 2x_1 + x_0)} = \omega^{16x_2}\,\omega^{8x_1}\,\omega^{4x_0}$$

$$= \underset{1}{\underset{\Downarrow}{\omega^{16x_2}}}\;\underset{1}{\underset{\Downarrow}{\omega^{8x_1}}}\;\underset{(\omega^4)^{x_0} = (-1)^{x_0}}{\underset{\Downarrow}{\omega^{4x_0}}}$$

$$\left(\frac{|0\rangle + \omega^{4x}|1\rangle}{\sqrt{2}}\right) = \frac{|0\rangle + (-1)^{x_0}|1\rangle}{\sqrt{2}}$$

FOR $x_0 = 0 \Rightarrow \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)$

FOR $x_0 = 1 \Rightarrow \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$

THIS IS A HADAMARD TRANSFORMATION

Middle (2nd) Factor for the 3 Qubit Example Construction

SECOND (MIDDLE) FACTOR

EXAMINE $\omega^{2x} = \omega^{2(4x_2 + 2x_1 + x_0)} = \omega^{8x_2}\omega^{4x_1}\omega^{2x_0}$

$= (\omega^8)^{x_2}(\omega^4)^{x_1}(\omega^2)^{x_0}$

$= (1)(-1)^{x_1}(i)^{x_0}$

$$\left(\frac{|0\rangle + \omega^{2x}|1\rangle}{\sqrt{2}}\right) = \left(\frac{|0\rangle + (-1)^{x_1}(i)^{x_0}|1\rangle}{\sqrt{2}}\right)$$

for $x_0 = 0 \Rightarrow H|x_1\rangle$

for $x_0 = 1$ $\left(\frac{|0\rangle + (-1)^{x_1}(i)^{x_0}|1\rangle}{\sqrt{2}}\right)$

When $x_0 = 1$ need an additional factor

IF APPLY $H|x_1\rangle \rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1 \\ -1^{x_1}\end{pmatrix}$ BUT IF

$x_0 = 1$ NEED $\frac{1}{\sqrt{2}}\begin{pmatrix}1 \\ -1^{x_1}\cdot i\end{pmatrix}$

PROBLEM TRANSFORM $\begin{pmatrix}1 \\ -1^{x_1}\end{pmatrix} \rightarrow \begin{pmatrix}1 \\ -1^{x_1}\cdot i\end{pmatrix}$

SOLN: INSERT 2x2 ROTATION MATRIX $R_1 = \begin{pmatrix}1 & 0 \\ 0 & i\end{pmatrix}$

$$\begin{pmatrix}1 & 0 \\ 0 & i\end{pmatrix}\begin{pmatrix}1 \\ -1^{x_1}\end{pmatrix} = \begin{pmatrix}1 \\ -1^{x_1}\cdot i\end{pmatrix}$$

$$\boxed{\text{FINAL RESULT} \begin{cases} x_0 = 0 & H|x_1\rangle \\ x_0 = 1 & R_1 H|x_1\rangle \end{cases}}$$

3rd (Least Significant) Factor for the 3 Qubit Example

THIRD FACTOR (LEAST SIGNIFICANT)

RIGHTMOST OUTPUT FACTOR HAS AN ω WITH NO EXPONENT IN THE NUMERATOR

$$\omega^{x} = \omega^{(4x_2 + 2x_1 + x_0)} = \omega^{4x_2}\,\omega^{2x_1}\,\omega^{x_0}$$

$$= (-1)^{x_2}\,(i)^{x_1}\,(\omega)^{x_0}$$

$$\left(\frac{|0\rangle + \omega^{x}|1\rangle}{\sqrt{2}}\right) \rightarrow \left(\frac{|0\rangle + (-1)^{x_2}(i)^{x_1}(\omega)^{x_0}|1\rangle}{\sqrt{2}}\right)$$

for $x_0 = 0$ $$= \frac{|0\rangle + (-1)^{x_2}(i)^{x_1}|1\rangle}{\sqrt{2}}$$

for $x_0 = 1$ $$= \frac{|0\rangle + (-1)^{x_2}(i)^{x_1}(\omega)^{x_0}|1\rangle}{\sqrt{2}}$$

3rd (Least Significant) Factor for the 3 Qubit Example (cont'd)

THIRD FACTOR (OBSERVATIONS)

WHEN $x_0 = 0$ THE LEAST SIGNIFICANT FACTOR REDUCES TO

$x_0 = 0$ $$\frac{|0\rangle + (-1)^{x_2}(i)^{x_1}|1\rangle}{\sqrt{2}}$$

COMPARE TO PREVIOUS CALCULATION [2^{nd} MIDDLE FACTOR]

$$\frac{|0\rangle + (-1)^{x_1}(i)^{x_0}|1\rangle}{\sqrt{2}}$$

- THIS IMPLIES IF $x_0 = 0$ APPLY SAME STEPS (PROCEDURE) TO $|x_2\rangle|x_1\rangle$ AS WAS USED FOR $|x_1\rangle|x_0\rangle$
- NOW HAVE TO ADJUST FOR CASE $x_0 = 1$. THE COLUMN VECTOR NEEDING ADJUSTMENT IS $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ (-1)^{x_2} \cdot (i)^{x_1} \end{pmatrix}$

HOWEVER IF $x_0 = 1$ ACTUAL RESULT SHOULD BE

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ (-1)^{x_2}(i)^{x_1}(\omega)^{x_0} \end{pmatrix}$$

- NEED TO CONSTRUCT ROTATION MATRICES TO TRANSFORM

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ (-1)^{x_2}(i)^{x_1} \end{pmatrix} \rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ (-1)^{x_2}(i)^{x_1}\omega^{x_0} \end{pmatrix}$$

- SOLUTION INSERT ANOTHER ROTATION MATRIX

$$R_2 = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}\begin{pmatrix} 1 \\ (-1)^{x_2}(i)^{x_1} \end{pmatrix} = \begin{pmatrix} 1 \\ (-1)^{x_2}(i)^{x_1}(\omega)^{x_0} \end{pmatrix}$$

3rd (Least Significant) Factor for the 3 Qubit Example (cont'd)

THIRD FACTOR (CONT'D)

FINAL RESULT WILL BE

$$\frac{|0\rangle + \omega^{x}|1\rangle}{\sqrt{2}} = \begin{cases} H\,|x_2\rangle & x_0 = 0 \quad x_1 = 0 \\ R_1\,H\,|x_2\rangle & x_0 = 0 \quad x_1 = 1 \\ R_2\,H\,|x_2\rangle & x_0 = 1 \quad x_1 = 0 \\ R_2 R_1 H\,|x_2\rangle & x_0 = 1 \quad x_1 = 1 \end{cases}$$

CONSTRUCT QUANTUM COMPUTING CIRCUIT BASED ON THESE MATHEMATICAL CALCULATIONS

HAVE 3 QUBITS

$|x_2\rangle$ ——

$|x_1\rangle$ ——

$|x_0\rangle$ ——

INSERT MATHEMATICAL FORMULATION 1 FACTOR AT A TIME

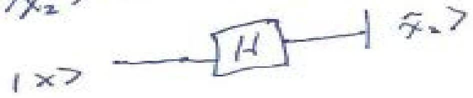
Building a Quantum Computing Circuit from the Mathematics of the Quantum Fourier Transform

QFT Quantum Gate Construction: 1st & 2nd Factor

FIRST FACTOR

BASED ON THE CALCULATION FROM FIRST FACTOR

$|\tilde{x}_2\rangle = H|x_0\rangle$



SECOND (MIDDLE) FACTOR

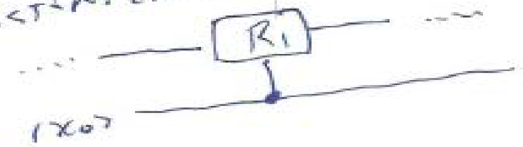
2 TERMS

$|\tilde{x}_2\rangle = H|x_0\rangle$

$$|\tilde{x}_1\rangle = \begin{cases} H|x_1\rangle & x_0 = 0 \\ R_1 H|x_1\rangle & x_0 = 1 \end{cases}$$

OBSERVATIONS

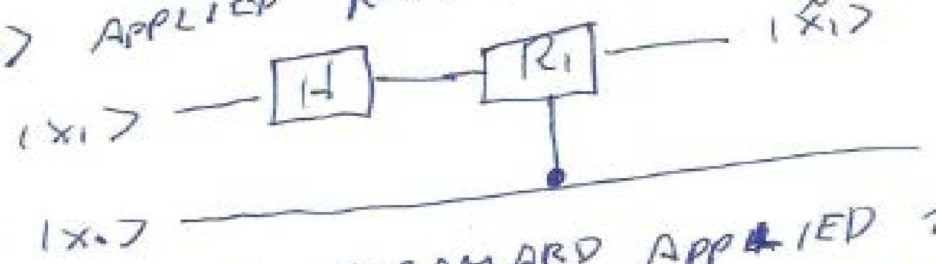
- APPLY H TO THE TWO LEAST SIGNIFICANT KETS $|x_0\rangle$ AND $|x_1\rangle$ UNCONDITIONALLY. H WILL ALWAYS BE USED IN THE COMPUTATION OF THE FINAL 2 MOST-SIGNIFICANT FACTORS
- CONDITIONALLY APPLY R_1 TO RESULT OF $H|x_1\rangle$ IF $x_0 = 1$
- APPLY TO LEAST SIGNIFICANT INPUT KETS $|x_1\rangle|x_0\rangle$ GET MOST SIGNIFICANT PORTION OF THE OUTPUTS FACTORIZATION (SWAPPING NEEDS TO BE INCLUDED)



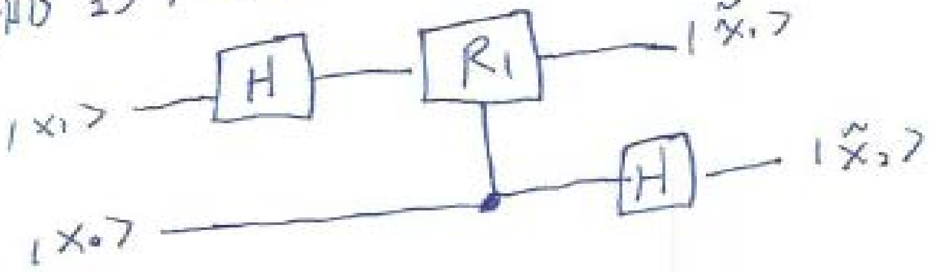
QFT Quantum Gate Construction: 2nd Factor (cont'd)

CIRCUIT CONSTRUCTION (CONT'D)

- APPLY UNCONDITIONAL HADAMARD GATE TO $|x_1\rangle$ BEFORE R_1
- $R_1 H |x_1\rangle$ APPLIED RIGHT TO LEFT



- INSERT ADDITIONAL HADAMARD APPLIED TO $|x_0\rangle$ ~~AS THE CONTROL~~
- **NOTE** ADD H AFTER x_0 USED TO CONTROL x_1's R_1 GATE
- IF THIS IS NOT DONE x_0 IS REPLACED BY HADAMARD



QFT Quantum Gate Construction: 3rd Factor

CIRCUIT CONSTRUCTION (CONTD)

LAST FACTOR (LEAST SIGNIFICANT)

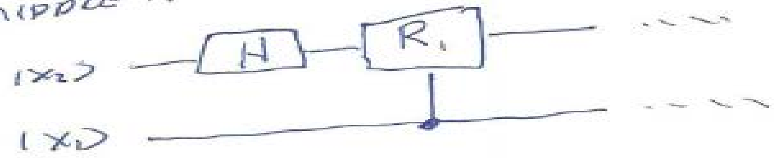
- RECALL $\frac{|0\rangle + w^{x}|1\rangle}{\sqrt{2}} = \frac{|0\rangle + (-1)^{x_2}(i)^{x_1}(\omega)^{x_0}|1\rangle}{\sqrt{2}}$

$= \frac{|0\rangle + (-1)^{x_2}(i)^{x_1}|1\rangle}{\sqrt{2}}$ $x = 0$

$\frac{|0\rangle + (-1)^{x_2}(i)^{x_1}(\omega)^{x_0}|1\rangle}{\sqrt{2}}$ $x = 1$

CASE $x_0 = 0$

- APPLY SAME CONSTRUCT TO $|x_2\rangle|x_1\rangle$ USED FOR $|x_1\rangle|x_0\rangle$ IN THE MIDDLE FACTOR



CASE $x_0 = 1$

STATE GENERATED WITH THIS CIRCUIT

$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ (-1)^{x_2} \cdot (i)^{x_1} \end{pmatrix}$

BUT WHAT IS NEEDED IS

$\begin{pmatrix} 1 \\ (-1)^{x_2}(i)^{x_1}(\omega)^{x_0} \end{pmatrix}$

QFT Quantum Gate Construction: 3rd Factor

QUANTUM COMPUTING
CIRCUIT CONSTRUCTION (CONT'D)

QFT Ex-9

- USE SAME TECHNIQUE APPLIED TO MIDDLE FACTOR CONSTRUCT A ROTATION MATRIX (CALL IT R_2)

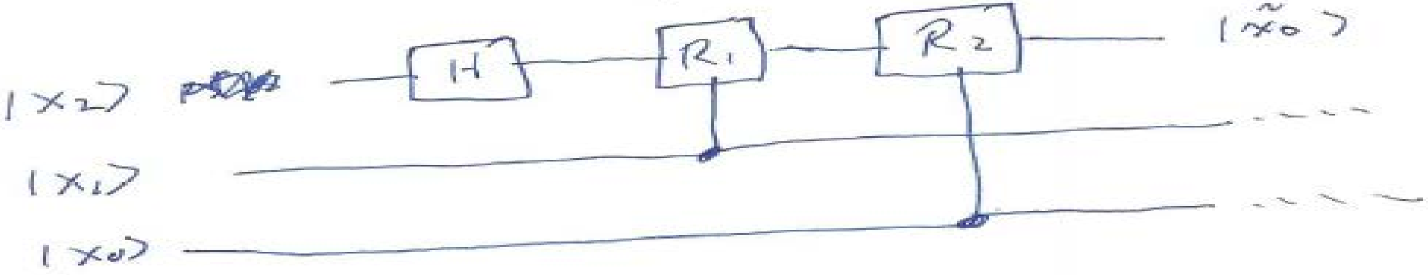
$$R_2 \equiv \begin{pmatrix}1 & 0\\ 0 & \omega\end{pmatrix}$$

$$\begin{pmatrix}1 & 0\\ 0 & \omega\end{pmatrix}\begin{pmatrix}1\\ (-1)^{x_2}(i)^{x_1}\end{pmatrix} =$$

$$= \begin{pmatrix}1\\ (-1)^{x_2}(i)^{x_1}(\omega)^{x_0}\end{pmatrix}$$

- COMPLETE CIRCUIT CONSTRUCT IS AS FOLLOWS

$$\frac{|0\rangle + \omega^{x}|1\rangle}{\sqrt{2}} = \begin{cases} H\,|x_2\rangle & x_0 = 0 \quad x_1 = 0\\ R_1\,H\,|x_2\rangle & x_0 = 0 \quad x_1 = 1\\ R_2\,H\,|x_2\rangle & x_0 = 1 \quad x_1 = 0\\ R_2\,R_1\,H\,|x_2\rangle & x_0 = 1 \quad x_1 = 1\end{cases}$$

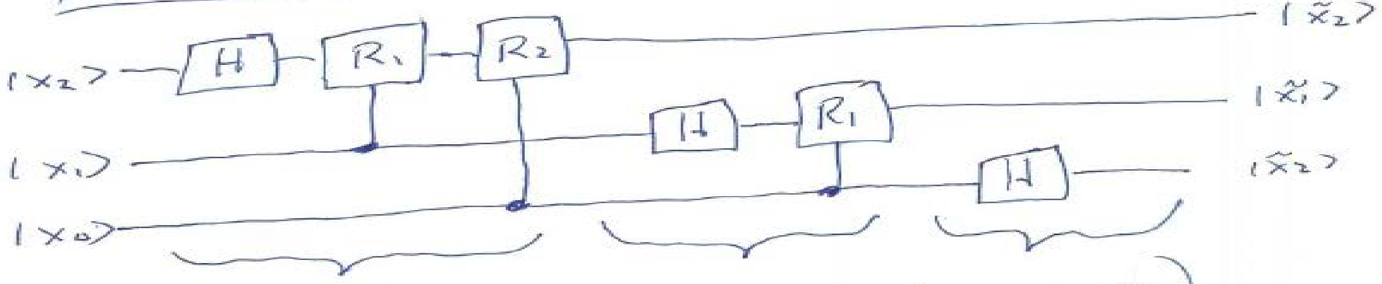


Full N[th] Order Qubit Quantum Fourier Transform Circuit

QUANTUM COMPUTING CIRCUIT CONSTRUCTION (CONTD)

QFT Ex-10

ASSEMBLE ALL FACTORS TO MAKE FULL CIRCUIT

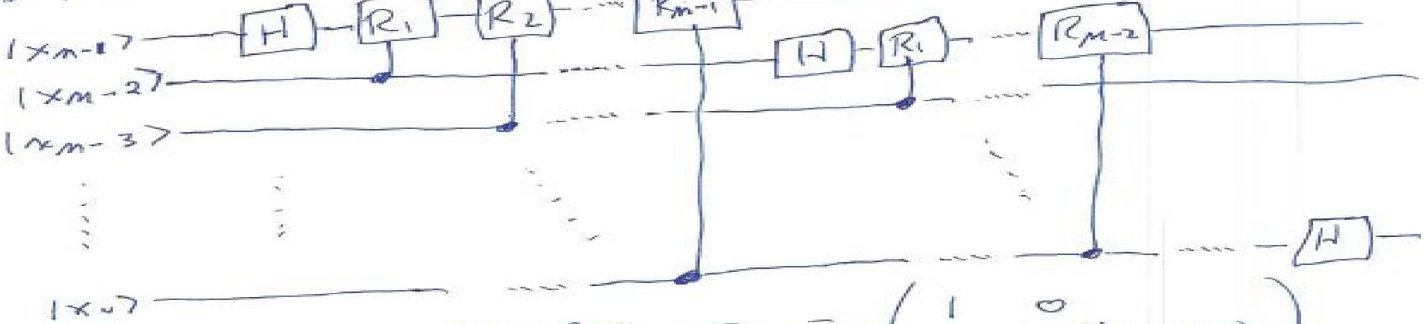


QFT CIRCUIT (N^{TH} ORDER)

- PATTERN IN THE INSERTED ROTATION GATES

$R_2 = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}$ $R_1 = \begin{pmatrix} 1 & 0 \\ 0 & \omega^2 \end{pmatrix}$ $H = \begin{pmatrix} 1 & 1 \\ 1 & \omega^4 \end{pmatrix}$

- FOR N QUBITS



GENERALIZE ROTATION MATRIX $R_K \equiv \begin{pmatrix} 1 & 0 \\ 0 & \omega^{2^{(m-k-1)}} \end{pmatrix}$

Last Slide